

# On measures of symmetry and floating bodies

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*Dedicated to Olek Pełczyński, a teacher and a friend*

## Abstract

We consider the following measure of symmetry of a convex  $n$ -dimensional body  $K$ :  $\rho(K)$  is the smallest constant for which there is a point  $x \in K$  such that for partitions of  $K$  by an  $n - 1$ -dimensional hyperplane passing through  $x$  the ratio of the volumes of the two parts is  $\leq \rho(K)$ . It is well known that  $\rho(K) = 1$  iff  $K$  is symmetric. We establish a precise upper bound on  $\rho(K)$ ; this recovers a 1960 result of Grünbaum. We also provide a characterization of equality cases (relevant to recent results of Nill and Paffenholz about toric varieties) and relate these questions to the concept of convex floating bodies.

**1. Introduction.** The structure of general  $n$ -dimensional (bounded) convex bodies is understood much less than that of the symmetric ones. For example, if we endow each of these classes with the appropriate version of the Banach-Mazur distance, then the asymptotic order (as  $n \rightarrow \infty$ ) of the diameter of the resulting compactum has been known for the symmetric case since the 1981 seminal Gluskin's paper [5], while the corresponding problem in the general case is wide open, with the first non-trivial results having been obtained only in the last several years [1, 11]. Various invariants have been proposed to explain the difference between these two classes; see, e.g., [9] for an early survey of related work. One such measure of symmetry (or rather of asymmetry) was mentioned to the author by Olek Pełczyński around the turn of the millennium. Given convex body  $K \subset \mathbb{R}^n$  and  $x \in K$ , consider all partitions of  $K$  into two parts by an  $n - 1$ -dimensional hyperplane  $H$  passing through  $x$ , and let  $\rho(K, x)$  be the largest ratio of volumes of the two parts. Next, let  $\rho(K) := \min_{x \in K} \rho(K, x)$ . Clearly  $\rho(K) = 1$  if  $K$  is centrally symmetric. (The reverse implication is also true but nontrivial, even for the larger class of star-shaped bodies; see [12] for details and [7] for additional references.) Olek's question was to establish a dimension-free upper bound on  $\rho(K)$ . I came up with an argument (which also established a precise upper bound on  $\rho(K)$  for  $K \subset \mathbb{R}^n$  and characterized bodies, for which that upper bound – call it  $\rho_n$  – is attained) and wrote it up some time afterwards, but then realized that the question had been considered and solved by Grünbaum in 1960 [8] and so I abandoned the project. However, it transpired very recently [2, 10] that Grünbaum's result and a characterization of  $K \subset \mathbb{R}^n$  for which  $\rho(K) = \rho_n$  were relevant to problems in toric geometry. Moreover, since the question in [8] was stated slightly differently than above, it led to an apparently non-equivalent

analysis of equality cases (see [8], Remark 4(i), p. 1260), less suitable – at least without additional work – for the applications considered in [10]. Accordingly, I am posting the manuscript (with added references and minor editorial changes).

**2. More background and the results.** The parameter  $\rho(K)$  is related to another geometrical concept, the convex floating bodies of  $K$ . [To give meaning to the formulae and to avoid artificial anomalies, it should be understood that  $K$  – and all bodies above and in what follows – is convex, compact and has a nonempty interior.] Let  $\delta \in (0, 1/2]$ ; slightly modifying the original definition from [14], let us denote by  $K^\delta$  the intersection of all half-spaces whose complements contain at most the proportion  $\delta$  of the volume of  $K$ . As is easy to see, if we call  $\phi(K)$  the largest number  $\delta$  for which the convex floating body  $K^\delta$  is nonempty, then  $\phi(K) = (\rho(K) + 1)^{-1}$ . The existence of a universal (i.e., independent of  $n$  and  $K$ ) strictly positive lower bound for  $\phi(K)$  (and, analogously, universal upper bound for  $\rho(K)$ ) has been a part of the folklore for some time [6, 15]. Here we prove the following “isometric” result.

**Theorem 1** *Let  $n \in \mathbb{N}$  and let  $K \subset \mathbb{R}^n$  be a convex, compact body with nonempty interior. Let  $c$  be the centroid of  $K$  and let  $H$  be an  $n - 1$ -dimensional hyperplane which passes through  $c$ , and thus divides  $K$  into two parts. Then the ratio of volumes of the two parts is  $\leq (1 + 1/n)^n - 1 =: \rho_n$  (which is  $< e - 1 < 1.7183$ ). Moreover, we have an equality iff  $K$  is a “pyramid,” i.e.,  $K = \text{conv}(\{v\} \cup B)$ , where  $B$  is an  $n - 1$ -dimensional convex body (the “base”) and  $v$  the vertex, and  $H$  is parallel to  $B$ .*

Similar statements about  $\rho(K)$  and  $\phi(K)$  are then simple consequences.

**Corollary 2** *In the notation and under the hypotheses of Theorem 1 we have*

- (i)  $\rho(K) \leq \rho_n$ ; moreover,  $\rho(K, c) \leq \rho_n$ , with equality iff  $K$  is a pyramid and the maximizing hyperplane  $H$  is parallel to a base of the pyramid.
- (ii)  $\phi(K) \geq (1 + 1/n)^{-n} =: \delta_n$ ; moreover,  $c \in K^{\delta_n}$  with  $c \in \partial K^{\delta_n}$  iff  $K$  is a pyramid.

The estimates  $\rho(K) \leq \rho_n$  and  $\phi(K) \geq \delta_n$  are, in general, best possible as seen from the example of a simplex.

**3. The proofs.** *Proof of Theorem 1* Without loss of generality we may assume that the centroid  $c$  of  $K$  is at the origin. Let  $\theta \in S^{n-1}$ ,  $H = \theta^\perp$  and consider the function  $f = f_\theta : \mathbb{R} \rightarrow \mathbb{R}^+$  defined by

$$f(t) := \text{vol}_{n-1}(K \cap (H + t\theta)). \quad (1)$$

It then follows that  $f$  is upper semi-continuous (since  $K$  is closed) and supported on some bounded interval  $[-a, b]$  with  $a, b > 0$ . (In fact it will follow from our arguments – and is likely well known – that the ratio  $|a|/|b| \in [1/n, n]$ .) Clearly,  $\text{vol}_n(K) = \int_{-a}^b f(t) dt$  and

$$\rho(K, c) = \max_{\theta \in S^{n-1}} \frac{\int_0^b f(t) dt}{\int_{-a}^0 f(t) dt}. \quad (2)$$

Additionally, the centroid being at the origin is equivalent to

$$\int_{-a}^b t f(t) dt = 0. \quad (3)$$

The only other property of  $f$  we shall need is that  $h := f^{1/(n-1)}$  is concave on  $[-a, b]$ , which is a consequence of the Brunn-Minkowski inequality. We note that the concavity implies continuity on  $(-a, b)$  and lower semi-continuity, hence continuity on  $[-a, b]$ . The Theorem will follow easily from the following two claims.

**Claim 3** *Let  $n \in \mathbb{N}$  and  $a, b > 0$ . For any  $\theta \in S^{n-1}$  and for any continuous function  $f : [-a, b] \rightarrow \mathbb{R}^+$  such that  $h := f^{1/(n-1)}$  is concave, there exists a closed convex body  $K \subset \mathbb{R}^n$  such that  $f$  is obtained from  $K, \theta$  via (1). If, additionally, (3) holds, then  $K$  may be chosen so that its centroid is at the origin. Finally,  $f$  is affine with  $f(-a) = 0$  (or  $f(b) = 0$ ) iff  $K$  is a pyramid and  $\theta$  is perpendicular to a base  $B$ ; in that case, if (3) also holds, the ratio from (2) equals  $(1 + 1/n)^n - 1$ .*

**Claim 4** *Let  $a, b > 0$ . Among continuous functions on  $[-a, b]$ , strictly positive on  $(-a, b)$ , verifying (3) and such that  $h := f^{1/(n-1)}$  is concave on  $[-a, b]$ , the largest value of the ratio appearing in (2) is achieved iff  $h$  is affine on  $[-a, b]$  with  $h(-a) = 0$ .*

Claim 3 shows that investigating  $\rho(K)$  and the ratio from (2) for functions verifying our assumptions are fully equivalent. Its proof is based on elementary geometric considerations. To construct  $K$  starting from  $f$ , we choose any  $n - 1$ -dimensional convex body  $B_0$  in  $H = \theta^\perp$  with  $\text{vol}_{n-1}(B_0) = 1$  and  $0 \in B_0$ , and set  $K := \bigcup_{t \in [-a, b]} t\theta + h(t)B_0$ . The “only if” part in the last assertion follows from the analysis of equality cases in the Brunn-Minkowski inequality (it occurs “essentially iff” the two sets are homothetic, see [12], Theorem 6.1.1). The details are left to the reader.

The proof of Claim 4 is also elementary, but less obvious. We will use the following lemma, variants of which exist in the literature. Similar arguments were employed (independently of this note) in [3], see also [4] for a more conceptualized application of closely related phenomena.

**Lemma 5** *Let  $M > 0$ ,  $m \in \mathbb{R}$  and  $n \in \mathbb{N}$ . We consider the set  $\mathcal{H}$  of functions  $h : \mathbb{R} \rightarrow \mathbb{R}^+$  which verify*

- (i) *the support of  $h$  is the interval  $[0, b]$  (for some  $b > 0$ )*
- (ii)  *$h$  is continuous and concave on  $[0, b]$*
- (iii)  *$h(0) = 1$  and the right derivative of  $h$  at 0 is  $\leq m$*
- (iv) *if  $f := h^{n-1}$ , then  $\int_0^b t f(t) dt = M$ .*

*The set  $\mathcal{H}$  is nonempty iff  $m \geq -1/\sqrt{Mn(n+1)}$ . In that case, set  $\mu(h) := \int_0^b f(t) dt$  for  $h \in \mathcal{H}$ . The minimal value of  $\mu(h)$  is attained iff  $h$  is affine on its support  $[0, b]$  with  $f(b) = 0$ . The maximal value of  $\mu(h)$  is attained iff  $h(t) = 1 + mt$  on the support of  $h$ .*

*Proof of Claim 4.* Since the ratio in (2) doesn't change if  $f(\cdot)$  is replaced by  $\alpha f(\beta \cdot)$ , it is enough to consider  $f$ 's whose support verifies  $[-1/n, 1/n] \subset [-a, b] \subset [-1, 1]$  and such that  $f(1) = 1$  (for the first inclusion, see the comments in the paragraph following (1) and at the very end of this note). Concavity of  $f^{1/(n-1)}$  gives then a lower bound on  $\int_{-a}^0 f(t) dt$  and an upper bound on  $\|f\|_\infty$  (both dependent on  $n$ ). It follows that the set of the functions  $f$  in question is compact (say, in the  $L_1$  metric, which is relevant here) and hence that the supremum of the ratio in (2) is attained. (This can also be proved in a variety of ways, including from the John's theorem.) Let  $f_0$  be such an extremal function (with support  $[-a_0, b_0]$ ), we shall show that it must be of the form indicated in Claim 4.

Indeed, if  $f_0$  was not affine on  $[-a_0, 0]$  with  $f(-a_0) = 0$ , we could apply the Lemma with  $h(\cdot) = f_0(-\cdot)^{1/(n-1)} : [0, a_0] \rightarrow \mathbb{R}^+$ ,  $M = -\int_{-a_0}^0 t f_0(t) dt$  and  $m$  equal to the right derivative of  $h$  at 0 to obtain an extremal  $h_1$  (supported on a possibly different interval  $[0, a_1]$ ) for which  $\mu(h_1) < \mu(h_0)$ . Defining  $f_1$  to coincide with  $f_0$  on  $[0, b_0]$  and with  $h_1(-\cdot)^{n-1}$  on  $[-a_1, 0]$  we would get a function for which the ratio from (2) was strictly larger than for  $f_0$ . At the same time, the conditions (iii) and (iv) from the Lemma (together with our choices of  $m, M$  would assure that, respectively,  $f_1^{1/(n-1)}$  was concave on  $[-a_1, b_0]$  and that (3) was satisfied. This shows that  $f_0(t) = (1 + t/a_0)^{n-1}$  for  $t \in [-a_0, 0]$ . A similar argument applied to  $h = f_0^{1/(n-1)} : [0, b_0] \rightarrow \mathbb{R}^+$ ,  $m = 1/a_0$  and the same  $M$  shows that  $f_0$  is affine on the entire interval  $[-a_0, b_0]$ , which concludes the proof of the Claim. (In fact we showed directly that affine  $h$ 's give the extremal value of the ratio from (2), so we didn't really need to know that the supremum was attained.)  $\square$

*Sketch of the proof of Lemma 5.* First, let us point out that the condition  $m \geq -1/\sqrt{Mn(n+1)}$  is equivalent to  $\mathcal{H} \neq \emptyset$ . Indeed, this is easily deduced from the observation that if  $h$  is supported on  $[0, b]$  and defined there by  $h(t) = 1 - t/b$ , then we have the relationship  $b = \sqrt{Mn(n+1)}$ . The assertion of the Lemma is then "essentially obvious from physical considerations." To obtain maximal "mass"  $\mu(h)$  for a given "moment"  $M$  (the constraint (iv)), we need to place the mass as closely to the axis  $t = 0$  as allowed by the concavity condition (ii) and by (iii). To minimize the mass for a fixed moment, we need to place the mass as far from  $t = 0$  as possible subject to (ii) and (iii). This is easily formalized. A very similar argument allows also to determine the largest possible value of  $b$  for a given  $M$ , and the smallest possible value for  $b$  (if at all possible) given  $m$  and  $M$ , and to subsequently deduce that the ratio of the two is at most  $n$ , thus implying the bounds on the ratio  $|a|/|b|$  stated in the paragraph following (1) and mildly used in the proof of Claim 4.  $\square$

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